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Turbulent Dissipation-Rate Equation for Compressible Flows

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The turbulent dissipation rate in a compressible flow is modeled in terms of its solenoidal and dilatational parts. In the k–ϵ framework the solenoidal dissipation rate is computed using the modeled dissipation-rate equation for incompressible flows. This is based on the assumption that the solenoidal dissipation rate for compressible flows follows the same dynamics as the dissipation rate for incompressible flows. We test this assumption by comparing the exact transport equations of these two quantities, both analytically and using direct numerical simulation data of a Mach 4 boundary layer. The two equations are found to be equivalent except for an additional term caused by the variation of fluid viscosity in the compressible case. This forms the rigorous basis for using the incompressible modeled dissipation-rate equation in a compressible boundary-layer flow provided the extra term caused by variation of fluid viscosity is included. Finally, the implications of the dissipation-rate analysis on the k–ω turbulence model are pointed out.

Introduction

THE turbulent dissipation rate is a quantity of fundamental interest in any turbulent flow because it determines the rate at which the turbulence decays. In two-equation turbulence models the dissipation rate or a related quantity is also used to define the turbulent length scale, which is used to model the eddy viscosity. In the k–ϵ model the turbulent dissipation rate ϵ is computed by solving a modeled form of its transport equation.

Initial attempts to obtain a model equation for the dissipation rate were heuristic. It was written by analogy with the turbulent kinetic energy equation and without any reference to the exact dissipation-rate equation. This was mainly because of scarcity of accurate data on which any rigorous modeling of the exact equation could be based. Lately, with the availability of direct numerical simulation (DNS) data for turbulent flows, the focus has shifted to the exact form of the dissipation-rate equation. Mansour et al.[1] identify the different terms in the ϵ-equation for incompressible flows based on the underlying physical mechanisms. Rodi and Mansour[2] and Nagano and Shimada[3] use DNS data to compute these terms and develop rigorous models for all of the terms in the incompressible dissipation-rate equation.

For homogeneous compressible flows the dissipation rate can be split into a solenoidal part and a dilatational part.[4,5] The solenoidal dissipation rate associated with the incompressible part of the turbulent motion is given by $\epsilon_s = \bar{\nu}\overline{\omega_i'\omega_i'}$, where ν is the kinematic viscosity and ω_i is the vorticity component in the x_i direction. Here, the overbar represents Reynolds averaging, and the prime denotes the fluctuations. The dilatational dissipation rate is defined as $\epsilon_d = \frac{4}{3}\bar{\nu}\overline{\theta'\theta'}$, where θ' is the fluctuating dilatation. In contrast to the solenoidal part, ϵ_d is identified as the dissipation rate associated with the dilatational or compressible part of the turbulent motion. For inhomogeneous turbulent flows Huang et al.[6] have shown that the dissipation rate consists of an inhomogeneous term and contributions caused by variation in fluid viscosity, in addition to ϵ_s and ϵ_d. In the k–ϵ model the inhomogeneous part of the dissipation rate and the contributions caused by the variation in fluid viscosity are neglected, whereas the dilatational part is modeled in terms of ϵ_s. DNS data[6,7] show that $\epsilon_d \ll \epsilon_s$ in wall-bounded compressible shear flows and that the models overpredict the value of ϵ_d.

In the k–ϵ model the solenoidal dissipation rate is computed by solving a modeled transport equation that is the same as the modeled ϵ equation for incompressible flows. This approach is universally accepted and yields satisfactory results in nonhypersonic boundary layers (up to Mach 5) in the absence of strong pressure gradient,[8] but it has very little rigorous basis. Specifically, there is no detailed study of how the transport equation for ϵ_s relates to the incompressible ϵ equation. In addition, there has not been enough research on the exact form of the transport equation for ϵ_s and how the modeled equation relates to the exact form. To the best of our knowledge, the only attempt in this regard is by Krishnamurthy and Shyy,[9] but their work is limited to the modeling of a few terms in the transport equation. Thus, there is a need for a comprehensive study of the exact equation for the solenoidal dissipation rate and to understand its connection to the incompressible dissipation rate equation. This is the focus of the current paper.

The exact transport equation for solenoidal dissipation rate in a compressible flow can be derived from the momentum equation. However, it is difficult to physically interpret the various terms. By comparison, the enstrophy $\overline{\omega_i'\omega_i'}$ equation is easier to understand. For incompressible flows Mansour et al.[1] use the relation between dissipation rate and enstrophy to identify the terms in the ϵ equation. Similarly, $\epsilon_s = \bar{\nu}\overline{\omega_i'\omega_i'}$ can be used to understand the ϵ_s equation in a compressible flow in terms of the corresponding enstrophy equation.

The paper is organized as follows. First, the enstrophy equations for incompressible and compressible flows are written, and the compressibility effects on enstrophy are identified. Next, the dissipation-rate equation is studied, where the incompressible ϵ equation is presented and its relation to the incompressible enstrophy equation is pointed out. For compressible flows the enstrophy equation is used to obtain an equation for ϵ_s. The different terms in the ϵ_s equation are identified, and a budget is computed using DNS data of a supersonic boundary layer.[10] Finally, modeling issues are addressed, where the relation between the ϵ_s equation and the incompressible dissipation-rate equation is analyzed. As noted earlier, this is the basis for modeling the ϵ_s equation. The similarities and differences between the two equations are discussed, followed by their modeling implications.

Enstrophy Equation

The transport equation for enstrophy in an incompressible flow is given by Tennekes and Lumley,[11] who described the physical mechanisms underlying different production terms in detail. To

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summarize,

$$\frac{D}{Dt}\overline{\omega_i'\omega_i'} = P_\omega^1 + P_\omega^2 + P_\omega^3 + P_\omega^4 + T_\omega + D_\omega - Y_\omega \tag{1}$$

where the left-hand side represents the material derivative of enstrophy and the first four terms on the right-hand side correspond to different production mechanisms. $P_\omega^1 = 2\bar{\omega}_j\overline{\omega_i' s_{ij}'}$ is the mixed production term, where $s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the rate of strain tensor and u_i is the velocity component in the x_i direction. $P_\omega^2 = 2\overline{\omega_i'\omega_j'}\bar{s}_{ij}$ is the production caused by mean rate of strain, $P_\omega^4 = 2\overline{\omega_i'\omega_j' s_{ij}'}$ is the turbulent production, and $P_\omega^3 = -2\overline{u_j'\omega_i'}\bar{\omega}_{i,j}$ is called the gradient production. $T_\omega = -(\overline{u_j'\omega_i'\omega_i'})_{,j}$ and $D_\omega = \nu(\overline{\omega_i'\omega_i'})_{,kk}$ represent the turbulent transport and the viscous diffusion, respectively, and $Y_\omega = 2\nu\overline{\omega_{i,j}'\omega_{i,j}'}$ is the viscous dissipation term.

For compressible flows the enstrophy equation can be written as

$$\frac{D}{Dt}\overline{\omega_i'\omega_i'} = P_\omega^1 + P_\omega^2 + P_\omega^3 + P_\omega^4 + T_\omega + D_\omega - Y_\omega + T_\omega^c + B_\omega + C_\omega \tag{2}$$

which is very similar to the equation presented by Robinson and Hassan.[12] The production terms P_ω^1, P_ω^2, and P_ω^4 are given by

$$P_\omega^1 = 2\bar{\omega}_j\overline{\omega_i' s_{ij}'} - 2\bar{\omega}_i\overline{\omega_i' u_{j,j}'}$$

$$P_\omega^2 = 2\overline{\omega_i'\omega_j'}\bar{s}_{ij} - 2\overline{\omega_i'\omega_i'}\bar{u}_{j,j}$$

$$P_\omega^4 = 2\overline{\omega_i'\omega_j' s_{ij}'} - 2\overline{\omega_i'\omega_i' u_{j,j}'} \tag{3}$$

The first part of the preceding source terms is identical to the corresponding production terms in Eq. (1) and are identified as the mixed production, the production caused by the mean rate of strain, and the turbulent production, respectively. The second part of each term is caused by nonzero dilatation in a compressible flow. The gradient production P_ω^3, the turbulent transport T_ω, the viscous diffusion D_ω, and the viscous dissipation Y_ω terms have the same expression as the corresponding terms in the incompressible case. The last three terms represent the effect of compressibility on enstrophy. $T_\omega^c = -\overline{\omega_i'\omega_i' u_{j,j}'}$ is the compressible transport term and can be interpreted as the transport of enstrophy into or out of a control volume caused by bulk compression or expansion. $B_\omega = -2\overline{\omega_i'(\nabla\rho^{-1}\times\nabla p)_i}$ represents the effect of baroclinic torques, and $C_\omega = 2\overline{\omega_i'(\nu'\omega_{i,jj} + \psi_\omega)}$ includes all of the effects caused by the spatial variation in viscosity of the fluid and its fluctuations. Here,

$$\psi_\omega = \frac{\partial\nu}{\partial x_k}\left(\frac{\partial\omega_i}{\partial x_k} - \frac{\partial\omega_k}{\partial x_i} - e_{ijk}\frac{4}{3}\frac{\partial^2 u_l}{\partial x_j\partial x_l}\right) + e_{ijk}\frac{\partial}{\partial x_j}\left[\frac{\mu_{,l}}{\rho}(u_{k,l} + u_{l,k}) - \frac{2\mu_{,k}}{3\rho}u_{l,l}\right] \tag{4}$$

where ρ is density, p is pressure, μ is molecular viscosity, and e_{ijk} is the alternating tensor.

Dissipation-Rate Equation

In an incompressible flow the dissipation rate of the turbulent kinetic energy is given by $\epsilon_I = 2\nu\overline{s_{ij}'s_{ij}'}$, which can be approximated by its isotropic form:

$$\epsilon_I \simeq \nu\overline{u_{i,j}'u_{i,j}'} \tag{5}$$

The exact form of the transport equation for ϵ_I is written in the form presented by Mansour et al.[1]:

$$\frac{D\epsilon_I}{Dt} = P_I^1 + P_I^2 + P_I^3 + P_I^4 + T_I + \Pi_I + D_I - Y_I \tag{6}$$

where the terms on the right-hand side are given as follows.

Mixed production:

$$P_I^1 = -\nu 2\overline{u_{i,j}'u_{k,j}'}\bar{u}_{i,k}$$

Production by mean rate of strain:

$$P_I^2 = -\nu 2\overline{u_{i,j}'u_{i,k}'}\bar{u}_{k,j}$$

Gradient production:

$$P_I^3 = -\nu 2\overline{u_{i,j}'u_k'}\bar{u}_{i,jk}$$

Turbulent production:

$$P_I^4 = -\nu 2\overline{u_{i,j}'u_{i,k}'u_{k,j}'}$$

Turbulent transport:

$$T_I = -\nu\overline{(u_k'u_{i,j}'u_{i,j}')}_{,k}$$

Pressure transport:

$$\Pi_I = -\nu 2\overline{(u_{i,j}'p_{,j}')}_{,i}/\rho$$

Viscous diffusion:

$$D_I = \nu^2(\overline{u_{i,j}'u_{i,j}'})_{,kk}$$

Viscous dissipation:

$$Y_I = \nu^2\, 2\,\overline{u_{i,jk}'u_{i,jk}'}$$

Here, subscript I denotes the incompressible flow quantities. The preceding nomenclature of the different terms is based on the fact that they are approximately equal to the corresponding terms in Eq. (1) times ν. The only exceptions are P_I^1 and P_I^2. We have

$$P_I^1 + P_I^2 = \nu\left(P_\omega^1 + P_\omega^2\right) \tag{7}$$

Although P_I^1 and P_I^2 are identified as the mixed production term and the production by mean rate of strain in Ref. 1, they together, not individually, represent these two production mechanisms. Note that the pressure transport term Π_I is absent in the enstrophy equation. Using DNS data in channel flows, Rodi and Mansour[2] show that Π_I has negligible contribution to the overall budget.

In a compressible flow the solenoidal dissipation rate is given by

$$\epsilon_s = \bar{\nu}\left(\overline{u_{i,j}'u_{i,j}'} - \overline{u_{i,j}'u_{j,i}'}\right) = \bar{\nu}\overline{\omega_i'\omega_i'} \tag{8}$$

The enstrophy equation (2) can, therefore, be transformed to an equation for ϵ_s:

$$\frac{D\epsilon_s}{Dt} = P_\epsilon^1 + P_\epsilon^2 + P_\epsilon^3 + P_\epsilon^4 + T_\epsilon + D_\epsilon - Y_\epsilon + T_\epsilon^c + B_\epsilon + C_\epsilon \tag{9}$$

where each term on the right-hand side, denoted by S, is related to the corresponding term in Eq. (2) as

$$S_\epsilon = \bar{\nu}S_\omega$$

except for

$$P_\epsilon^1 + P_\epsilon^2 = \bar{\nu}\left(P_\omega^1 + P_\omega^2\right) \tag{10}$$

which is identical to the incompressible case [see Eq. (7)]. Also,

$$C_\epsilon = \bar{\nu}C_\omega + \overline{\omega_i'\omega_i'}\frac{D\bar{\nu}}{Dt}$$

The expressions for all of the terms in Eq. (9) are given next. Each term represents the same physical process as the corresponding term in the enstrophy equation (2).

Mixed production:

$$P_\epsilon^1 = -\bar{\nu}2\overline{(u_{i,j}' - u_{j,i}')u_{k,j}'}\bar{u}_{i,k}$$

Production by mean rate of strain:

$$P_\epsilon^2 = -\bar{\nu}2\overline{(u_{i,j}' - u_{j,i}')u_{i,k}'}\bar{u}_{k,j}$$

Gradient production:

$$P_\epsilon^3 = -\bar{\nu}2\overline{(u'_{i,j} - u'_{j,i})u'_k}\bar{u}_{i,jk}$$

Turbulent production:

$$P_\epsilon^4 = -\bar{\nu}2\overline{(u'_{i,j} - u'_{j,i})u'_{i,k}u'_{k,j}}$$

Turbulent transport:

$$T_\epsilon = -\bar{\nu}\overline{[u'_k(u'_{i,j} - u'_{j,i})u'_{i,j}]}_{,k}$$

Viscous diffusion:

$$D_\epsilon = \bar{\nu}^2\left(\overline{u'_{i,j}u'_{i,j} - u'_{i,j}u'_{j,i}}\right)_{,kk}$$

Viscous dissipation:

$$Y_\epsilon = \bar{\nu}^2 2\overline{(u'_{i,j} - u'_{j,i})_{,k}u'_{i,jk}}$$

Compressible transport term:

$$T_\epsilon^c = \bar{\nu}\overline{(u'_{i,j} - u'_{j,i})u'_{i,j}u'_{k,k}}$$

Baroclinic term:

$$B_\epsilon = \bar{\nu}2\overline{(u'_{i,j} - u'_{j,i})\rho_{,j}p_{,i}/\rho^2}$$

Viscosity variation term:

$$C_\epsilon = \bar{\nu}2\overline{(u'_{i,j} - u'_{j,i})(\nu' u_{i,kkj} + \psi_\epsilon)} + \frac{\epsilon_s}{\bar{\nu}}\frac{D\bar{\nu}}{Dt}$$

where

$$\psi_\epsilon = \frac{\partial \nu}{\partial x_j}\left(u_{i,kk} + \frac{1}{3}u_{k,ik}\right) + \frac{\partial}{\partial x_j}\left[-\frac{\mu_{,k}}{\rho}(u_{i,k} + u_{k,i}) + \frac{2\mu_{,i}}{3\rho}u_{k,k}\right]$$

The terms P_ϵ^1, P_ϵ^2, and P_ϵ^4 are identified as the mixed production term, the production caused by the mean rate of strain, and the turbulent production term, respectively, but they also include the effect of the bulk compression as identified in Eq. (3). In turbulent boundary layers at nonhypersonic Mach numbers, the effect of bulk compression is negligible, as shown later in the budget of the ϵ_s equation in a Mach 4 boundary layer. However, this might not be true in the presence of strong compression in a flow, for example, as a result of a shock wave.

A comparison of Eqs.(6) and (9) shows that both equations have terms representing the four production mechanisms, turbulent transport, viscous diffusion, and viscous dissipation. These are referred to as the "common terms" hereafter. In each common term $u'_{i,j}$ in the incompressible case is replaced by $u'_{i,j} - u'_{j,i}$ in the compressible equation. The last three terms in the ϵ_s equation are absent in the incompressible equation, and they represent additional effects of compressibility on ϵ_s. The pressure transport term in Eq. (6) is absent in the compressible case.

A budget of the solenoidal dissipation rate is computed using DNS data of a Mach 4 boundary layer[10] on an adiabatic wall at a Reynolds number based on momentum thickness Re_θ of 7×10^3. The Reynolds number based on the friction velocity Re_τ is 8×10^2. The DNS uses a third-order-accurate, high-bandwidth, weighted essentially nonoscillatory (WENO) scheme.[13] The scheme was designed to give accurate results for compressible DNS by using additional points in the computational stencil to reduce dispersion and dissipation error. In addition, the fluxes are constructed with symmetric data except in regions of large gradient, where upwind-biased data are used. Thus, the scheme has essentially zero dissipation except at high wave number, where a very low dissipation is added. The result is a highly accurate and robust scheme for compressible DNS.

The dimensions of the computational domain are 6.5δ, 1.6δ, and 11δ in the streamwise, spanwise, and wall normal directions, where δ is the boundary-layer thickness. The number of grid points in these directions are 384, 256, and 128, respectively. The Van-Driest transformed velocity profile matches the law of the wall ($u^+ = 2.44 \log y^+ + 5.5$), and the two-point correlations and energy spectra confirm the validity of the simulations. In addition, the turbulent kinetic energy budget shows good agreement with the Mach 2.5 boundary-layer simulation of Guarini et al.[7]

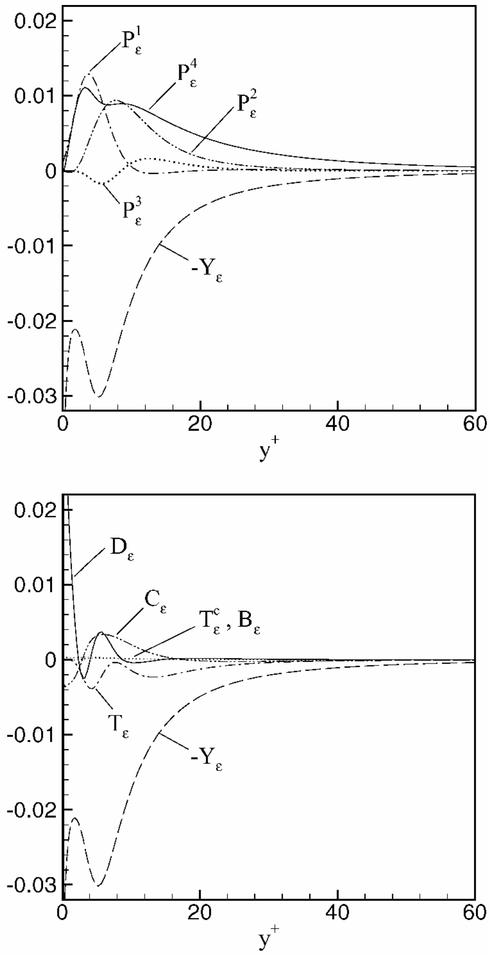

Fig. 1 Budget of the ϵ_s equation computed using DNS data in a Mach 4 boundary layer. All of the terms are normalized by $u_\tau^6/\bar{\nu}_w^2$.

The magnitude of the different terms in the ϵ_s equation is shown in Fig. 1. Following the work by Rodi and Mansour[2] for incompressible flows, all of the terms are normalized by $u_\tau^6/\bar{\nu}_w^2$, where u_τ is the friction velocity and ν_w is the kinematic viscosity at the wall. It can be seen that only the turbulent production term P_ϵ^4 and the viscous dissipation term Y_ϵ are significant in the high-Reynolds-number region away from the wall ($y^+ > 40$), and they approximately balance each other. Here, $y^+ = yu_\tau/\nu_w$ is a normalized distance from the wall. Closer to the wall, P_ϵ^1 and P_ϵ^2 are as large as P_ϵ^4, whereas P_ϵ^3 is relatively small. This pattern of the production terms is very similar to the data presented by Rodi and Mansour.[2] In the near-wall region the dissipation term Y_ϵ is large to balance all of the production terms, whereas the viscous diffusion term D_ϵ balances the dissipation at the wall. D_ϵ is small compared to the production terms in the rest of the boundary layer. The magnitude of the turbulent transport term T_ϵ is of the same order as that of the viscous diffusion term, except for very close to the wall, where D_ϵ is large. The compressible transport term T_ϵ^c and the baroclinic term B_ϵ are negligible compared to the production terms, whereas the source term caused by variation in fluid viscosity C_ϵ is relatively large in the near-wall region.

C_ϵ consists of terms of different types, which can be collected into three groups,

$$C_\epsilon = C_{\epsilon,0} + C_{\epsilon,1} + C_{\epsilon,2}$$

where $C_{\epsilon,0}$ represents the contributions of the fluctuations in fluid viscosity and $C_{\epsilon,1}$ and $C_{\epsilon,2}$ denote the terms containing the first and

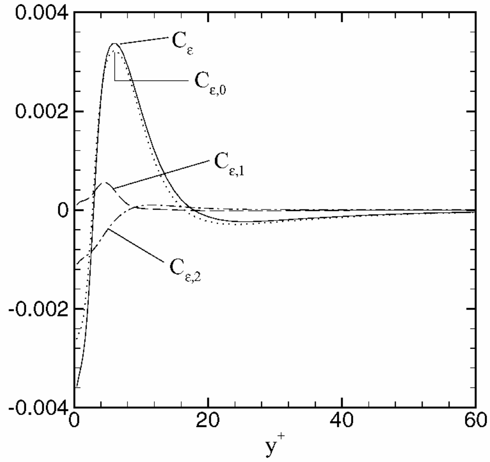

Fig. 2 Contribution to C_ϵ caused by the fluctuations in fluid viscosity and the mean viscosity gradients in a Mach 4 boundary layer. All of the quantities are normalized by $u_\tau^6/\bar{\nu}_w^2$.

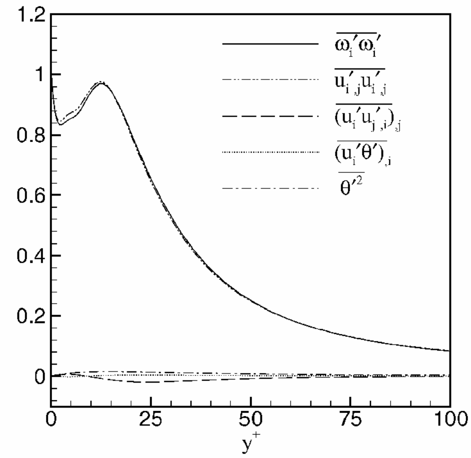

Fig. 3 Comparison of the terms in Eq. (11) as computed using the Mach 4 boundary-layer data. All of the quantities are normalized by the value of enstrophy at the wall.

second derivatives of the mean viscosity, respectively. The expressions for the three parts of C_ϵ are given next:

$$C_{\epsilon,0} = 2\bar{\nu}\Big[\overline{\nu' u'_{i,j} u_{i,jkk}} + \overline{\nu'_{,j} u'_{i,j} u_{i,kk}} + 2\overline{\mu'_{,jk} u'_{i,j} s_{ik}/\rho} + 2\overline{\mu'_{,k} u'_{i,j}(s_{ik}/\rho)_{,j}}\Big]$$

$$C_{\epsilon,1} = 2\bar{\nu}\Big[\overline{u'_{i,j}u'_{i,kk}}\bar{\nu}_{,j} + 2\overline{u'_{i,j}(s_{ik}/\rho)_{,j}}\bar{\mu}_{,k}\Big] + \overline{u'_{i,j}u'_{i,j}}\bar{u}_k\bar{\nu}_{,k}$$

$$C_{\epsilon,2} = 4\bar{\nu}\overline{u'_{i,j}s_{ik}/\rho}\bar{\mu}_{,jk}$$

where the remaining terms in C_ϵ are neglected because their contributions are small compared to the terms just presented. Figure 2 shows the three parts of C_ϵ as computed using the boundary-layer data. The viscosity fluctuations term $C_{\epsilon,0}$ has a dominant contribution, and $C_{\epsilon,1}$ and $C_{\epsilon,2}$ are relatively small.

Modeling of Solenoidal Dissipation Rate

The solenoidal dissipation rate in a compressible flow is computed using a modeled transport equation, which is not derived from its exact form (9); rather the modeled dissipation-rate equation for incompressible flows is directly used in the compressible case. This is based on the assumption that ϵ_s follows the same dynamics as the dissipation rate in an incompressible flow. In other words, it is assumed that the transport equations for ϵ_s and ϵ_I have the same form. To check the validity of this assumption, we compare the expressions for ϵ_s and ϵ_I, and then compare their transport equations, both analytically and with the aid of the DNS data.[10]

The solenoidal dissipation rate is proportional to the enstrophy, and the enstrophy can be written as

$$\overline{\omega'_i\omega'_i} = \overline{u'_{i,j}u'_{i,j}} - \overline{u'_{i,j}u'_{j,i}}$$

$$= \overline{u'_{i,j}u'_{i,j}} - \frac{\partial}{\partial x_j}\overline{u'_i u'_{j,i}} - \frac{\partial}{\partial x_i}\overline{u'_i\theta'} + \overline{\theta'^2} \tag{11}$$

The last three terms represent the contributions of inhomogeneity, the velocity–dilatation correlation, and compressibility, respectively. Figure 3 shows the magnitude of the terms in the preceding equation computed using the boundary-layer data.[10] We see that $\overline{\theta'^2} \ll \overline{\omega'_i\omega'_i}$, which is similar to the results obtained by Huang et al.[6] for channel flows with cooled walls and for an adiabatic wall boundary layer by Guarini et al.[7] Also, the second and the third terms in Eq. (11) have negligible contribution to the enstrophy. The last three terms constitute $\overline{u'_{j,i}u'_{i,j}}$, and therefore

$$\overline{u'_{j,i}u'_{i,j}} \ll \overline{u'_{i,j}u'_{i,j}} \tag{12}$$

Thus, ϵ_s can be split into two parts: the first is identical to ϵ_I [see Eqs. (5) and (8)], and the second term represents the effect of inhomogeneity and compressibility. Equation (12) shows that the second part of ϵ_s is negligible compared to the first, and therefore

$$\epsilon_s \simeq \bar{\nu}\overline{u'_{i,j}u'_{i,j}} \tag{13}$$

Thus, ϵ_s is equivalent to ϵ_I in a compressible flow.

Next, the preceding comparison of ϵ_s and ϵ_I is extended to the common terms in the transport equation (9), which have a corresponding term in the ϵ_I equation. Each term is split into two parts: the first part is identical to the respective incompressible term and the second part represents the effect of inhomogeneity and compressibility:

$$P_\epsilon^1 = P_I^1 + 2\bar{\nu}\overline{u'_{j,i}u'_{k,j}}\bar{u}_{i,k}, \qquad P_\epsilon^2 = P_I^2 + 2\bar{\nu}\overline{u'_{j,i}u'_{i,k}}\bar{u}_{k,j}$$

$$P_\epsilon^3 = P_I^3 + 2\bar{\nu}\overline{u'_{j,i}u'_k}\bar{u}_{i,jk}, \qquad P_\epsilon^4 = P_I^4 + 2\bar{\nu}\overline{u'_{j,i}u'_{i,k}u'_{k,j}}$$

$$T_\epsilon = T_I + \bar{\nu}\big(\overline{u'_k u'_{j,i}u'_{i,j}}\big)_{,k}, \qquad D_\epsilon = D_I - \bar{\nu}^2\big(\overline{u'_{i,j}u'_{j,i}}\big)_{,kk}$$

$$Y_\epsilon = Y_I - 2\bar{\nu}^2\overline{u'_{j,ik}u'_{i,jk}} \tag{14}$$

The preceding terms are computed using the boundary-layer data, and the results are shown in Fig. 4. It is found that the second part of each term is negligible compared to the first part. This is analogous to the result obtained in Eq. (12). Thus, the transport equation for solenoidal dissipation can be simplified to

$$\frac{D\epsilon_s}{Dt} = P_I^1 + P_I^2 + P_I^3 + P_I^4 + T_I + D_I - Y_I + C_\epsilon \tag{15}$$

where the compressible transport term T_ϵ^c and the baroclinic term B_ϵ are neglected. These terms have a negligible contribution to the overall budget (see Fig. 1). Comparison of the simplified ϵ_s equation (15) with the incompressible dissipation-rate equation (6) shows that the right-hand sides are identical except for the terms Π_I and C_ϵ. The pressure transport term Π_I in the incompressible equation is found to be small by Rodi and Mansour,[2] whereas the effect of the viscosity variation term C_ϵ is not negligible (see Fig. 1). Thus, ϵ_s follows the same transport equation as ϵ_I except for C_ϵ. This provides a rigorous basis for the use of the incompressible modeled ϵ equation in compressible wall-bounded shear flows. However, the effect of C_ϵ needs to be included in the model equation.

In the k–ω model the specific dissipation rate $\hat{\omega}$ is defined as[8]

$$\hat{\omega} = \epsilon/\beta^* k \tag{16}$$

where k is the turbulent kinetic energy, ϵ is the turbulent dissipation rate, and β^* is a model constant. In a compressible boundary layer $\epsilon \simeq \epsilon_s$, and therefore a transport equation for $\hat{\omega}$ can be derived by

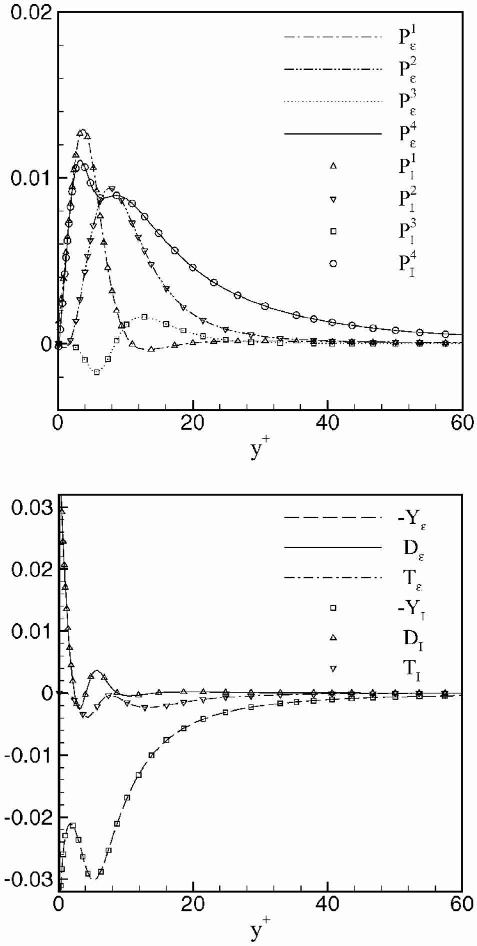

Fig. 4 Different terms in the ϵ_s equation along with the corresponding incompressible terms (symbols) given in Eq. (14) as computed using DNS data of a Mach 4 boundary layer.

combining the equations for k and ϵ_s. The compressible terms in the k equation are negligible in wall-bounded compressible shear flows.[6,7] Also, the preceding analysis shows that the only additional term in the ϵ_s equation, as compared to the incompressible equation, is C_ϵ. Therefore, the transport equation for $\hat{\omega}$ is identical to the corresponding incompressible equation, except for a viscosity variation term. This justifies the application of incompressible modeled $\hat{\omega}$ equation in compressible boundary-layer flows, but the source term caused by the variation of fluid viscosity should be modeled.

Conclusions

In this paper we study the turbulent dissipation-rate equation for compressible flows. In wall-bounded shear layers the dissipation is almost entirely caused by its solenoidal part. We derive a transport equation for the solenoidal dissipation rate ϵ_s based on its relation with the enstrophy in the flow and compute a budget of the equation using DNS data of a Mach 4 boundary layer. The modeling of the ϵ_s equation in the k–ϵ framework is based on the incompressible dissipation rate ϵ_I equation. Comparison of the two equations, analytically and with the aid of DNS data, shows that they are identical except for a source term caused by the variation of fluid viscosity in the compressible case. This close relation between the ϵ_s and ϵ_I equations forms the rigorous basis for the application of the incompressible model equation in compressible boundary layers. The extension of the result to the k–ω model is also discussed.

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